

Closest Point on a Plane

Problem. Determine the point $P = (x, y, z)$ in the plane

$$ax + by + cz = d$$

that is closest to the point $P_0 = (x_0, y_0, z_0)$ not on the plane.

It is enough to minimize the square of the distance, since squaring does not change where the minimum occurs:

$$D^2(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2.$$

Thus

$$\nabla D^2 = \langle 2(x - x_0), 2(y - y_0), 2(z - z_0) \rangle.$$

The constraint is

$$g(x, y, z) = ax + by + cz - d = 0, \quad \nabla g = \langle a, b, c \rangle = \mathbf{n}.$$

By the method of Lagrange multipliers,

$$\nabla D^2 = \lambda \nabla g,$$

so

$$2(x - x_0) = \lambda a, \quad 2(y - y_0) = \lambda b, \quad 2(z - z_0) = \lambda c.$$

Therefore

$$x = x_0 + \frac{\lambda a}{2}, \quad y = y_0 + \frac{\lambda b}{2}, \quad z = z_0 + \frac{\lambda c}{2}.$$

Since P lies in the plane, substitute these expressions into $ax + by + cz = d$:

$$\begin{aligned} d &= a \left(x_0 + \frac{\lambda a}{2} \right) + b \left(y_0 + \frac{\lambda b}{2} \right) + c \left(z_0 + \frac{\lambda c}{2} \right) \\ &= ax_0 + by_0 + cz_0 + \frac{\lambda}{2}(a^2 + b^2 + c^2). \end{aligned}$$

Writing $\mathbf{p}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{n} = \langle a, b, c \rangle$, this becomes

$$d = \mathbf{n} \cdot \mathbf{p}_0 + \frac{\lambda}{2} \|\mathbf{n}\|^2,$$

and hence

$$\frac{\lambda}{2} = \frac{d - \mathbf{n} \cdot \mathbf{p}_0}{\|\mathbf{n}\|^2}.$$

Substituting back into the coordinates gives the compact vector formula

$$\boxed{\mathbf{p} = \mathbf{p}_0 + \frac{d - \mathbf{n} \cdot \mathbf{p}_0}{\|\mathbf{n}\|^2} \mathbf{n}}$$

for the point on the plane closest to P_0 .

Normalized form. If we normalize the normal vector, define

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}.$$

Dividing the plane equation by $\|\mathbf{n}\|$ gives

$$\hat{\mathbf{n}} \cdot \mathbf{p} = \hat{d}, \quad \hat{d} = \frac{d}{\|\mathbf{n}\|}.$$

Because $\hat{\mathbf{n}}$ is a unit vector, $\hat{d}$ is the signed distance from the origin to the plane. Also, $\hat{\mathbf{n}} \cdot \mathbf{p}_0$ is the scalar projection of $\mathbf{p}_0$ onto the normal direction.

The closest-point formula therefore simplifies to

$$\boxed{\mathbf{p} = \mathbf{p}_0 - (\hat{\mathbf{n}} \cdot \mathbf{p}_0 - \hat{d})\hat{\mathbf{n}}}.$$

The scalar $\hat{\mathbf{n}} \cdot \mathbf{p}_0 - \hat{d}$ is the signed perpendicular displacement from the plane to P_0 ; subtracting that displacement moves P_0 directly onto the plane.